An Analysis of an Aluminum Cantilever

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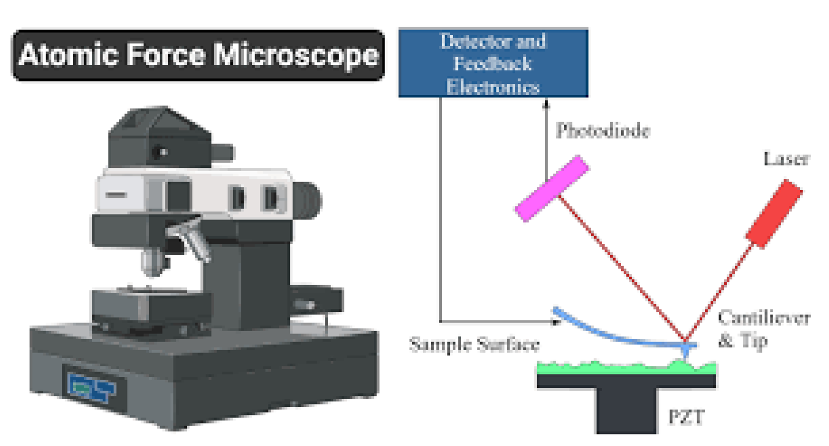
Abstract

A cantilever is a thin beam that is fixed at one end. The end that is not fixed is free to oscillate. This analysis involved a single aluminum cantilever. A PASCO 750 interface and a motion sensor II were used to record the motion of the cantilever. The data was analyzed to find the frequency and modes of oscillation of the beam. A stopwatch can be used for lower frequencies. The frequency of oscillation is inversely proportional to the square of the cantilever's length and independent of amplitude. This relation led to a dynamic calculation of flexural rigidity. We extended the analysis to include a static calculation for the flexural rigidity using Young's modulus and the second moment of area along with other means of ensuring an intensive self-consistent analysis.

Introduction

The cantilever is used in many applications as far apart as construction to microelectromechanical systems. A couple examples are a cantilever barn and an atomic force microscope. They are shown below:



Basic Theory

The Euler-Lagrange equation for a dynamic beam is:

$$S = \int_{t_0}^{t}\int_{0}^{L}\left[\frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{EI}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + q(x)u(x)\right] dxdt \qquad \textbf{(EQ.1)}$$

Where ρ is linear mass density, E is Young's modulus, I is the second moment of area, u is the deflection of the beam, and q is the distributed load. The Euler-Lagrange equation is used to determine the function that minimizes the functionless S. For a classical dynamic beam, the Euler-Lagrange equation becomes:

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right) = -\rho\frac{\partial^2 u}{\partial t^2} + q(x) \qquad \textbf{(EQ.2)}$$

In our case, E and I are independent of position, x, and there is no external load, so the equation becomes:

$$EI\frac{\partial^4 u}{\partial x^4} = -\rho\frac{\partial^2 u}{\partial t^2} \qquad \textbf{(EQ.3)}$$

This equation helps determine the deflection of the beam at a position x and time t. This equation can be solved by Fourier decomposition of the displacement into the sum of harmonic value:

$$u(x,t) = \mathrm{Re}\left[\hat{u}(x)e^{-i\omega_n t}\right] \qquad \textbf{(EQ.4)}$$

where ω_n is the natural frequency of vibration. This gives us the ordinary differential equation:

$$EI\frac{d^4\hat{u}}{dx^4} - \rho\omega_n^2\hat{u} = 0 \qquad \textbf{(EQ.5)}$$

The general solution of this equation is:

$$\hat{u} = A_0\cosh(\beta_n L) + A_1\sinh(\beta_n L) + A_2\cos(\beta_n L) + A_3\sin(\beta_n L) \qquad \textbf{(EQ.6)}$$

where:

$$\beta_n = \sqrt[4]{\left(\frac{\rho\omega_n^2}{EI}\right)} \qquad \textbf{(EQ.7)}$$

Boundary considerations:
Since the beam equation contains a fourth order derivative of x, we need four boundary conditions to find a unique solution $u(x,t)$. At the fixed end of the beam there can be no displacement or rotation of the beam. This means at the fixed end both deflection and slope are zero. At the free end since there is no external bending moment applied the bending moment is zero. Also since there is no external force applied, the shear force at the free end is zero.

Basic Theory Continued

This gives us two boundary conditions for the fixed end:

$$\hat{u}|_{x=0} = 0 \qquad \textbf{(EQ.8)}$$

$$\frac{\partial\hat{u}}{\partial x}|_{x=0} = 0 \qquad \textbf{(EQ.9)}$$

and two conditions for the free end:

$$\frac{\partial^2\hat{u}}{\partial x^2}|_{x=L} = 0 \qquad \textbf{(EQ.10)}$$

$$\frac{\partial^3\hat{u}}{\partial x^3}|_{x=L} = 0 \qquad \textbf{(EQ.11)}$$

If we apply these conditions, non-trivial solutions are found when:

$$\cosh(\beta_n L)\cos(\beta_n L) + 1 = 0 \qquad \textbf{(EQ.12)}$$

The first few modes are:

$$\beta_1 L = 1.875,\ \beta_2 L = 4.694,\ and\, \beta_3 L = 7.855$$

Each displacement solutions is called a mode and the shape of the displacement curve is called the mode shape. The corresponding frequencies of vibration for the modes are determined by:

$$\omega_n = \beta_n^2\sqrt{\frac{EI}{\rho}} \qquad \textbf{(EQ.13)}$$

These boundary conditions, along with two new ones, can be used to determine the mode shapes from the solution for the displacement. The new initial conditions are when t=0 for both velocity and displacement of the beam:

$$\hat{u}|_{t=0} = p(x) \qquad \textbf{(EQ.14)}$$

$$\frac{\partial\hat{u}}{\partial t}|_{t=0} = v(x) \qquad \textbf{(EQ.15)}$$

Where $p(x)$ is the initial displacement and $v(x)$ is the initial velocity of the beam. Using separation of variables, the periodic solution can be written as a product of position and time functions. This is done by letting the response equal the product of the independent position and time functions, and setting them equal to a positive separation constant:

$$u(x,t) = F(x)T(t) \qquad \textbf{(EQ.16)}$$

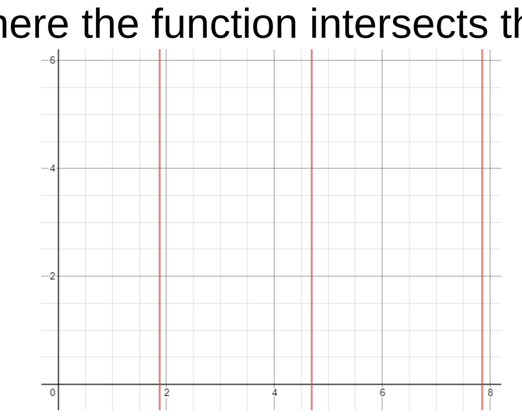
$F(x)$ becomes:

$$F(x) = C\big((\sin\beta_n L + \sinh\beta_n L)(\cosh\beta_n x - \cos\beta_n x) + (\cosh\beta_n L + \cos\beta_n L)(\sin\beta_n x - \sinh\beta_n x)\big) \qquad \textbf{(EQ.17)}$$

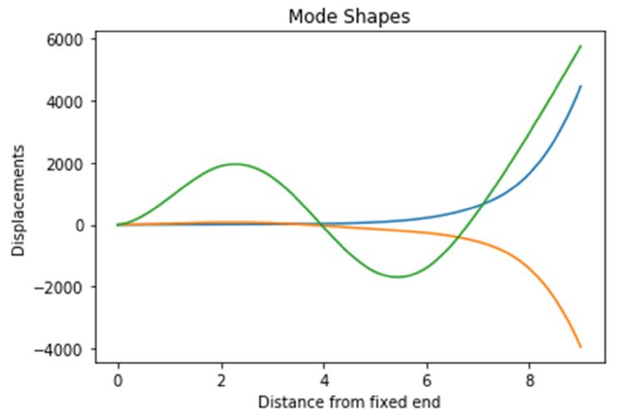
Where C is an arbitrary constant which is normally complex. This function determines the mode shapes. When graphing mode shapes C=1 is often chosen. The function with respect to time becomes:

$$T(t) = A_n\cos\omega_n t + B_n\sin\omega_n t \qquad \textbf{(EQ.18)}$$

From graphing equation 12, the mode solutions can be graphically shown. The first three modes are where the function intersects the x-axis:



The first three mode shapes(Mode 1 blue, mode 2 yellow, mode 3 green):



Cantilever Specs

Length, l (m)	Width, w (m)	Thickness, t(m)	Mass, m (kg)
0.9165	0.0382	0.0033	0.1945

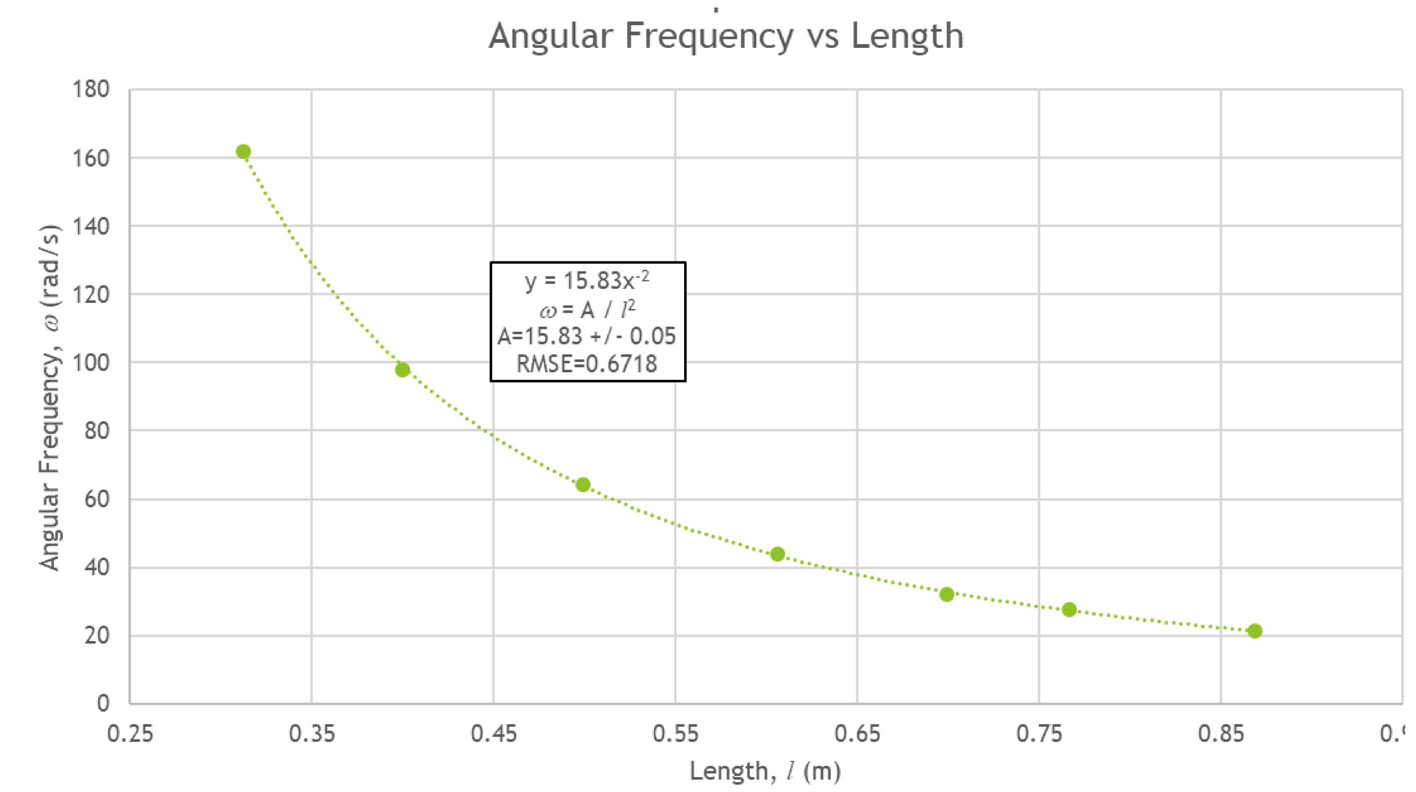
Experiment

With one end of the cantilever fixed, the frequency was measured for various lengths. A stopwatch can be used but since the frequency is inversely proportional to the square of the cantilever's length it is easier to do at longer lengths. The position versus time is then graphed and the angular frequency is found via a fit or a fast Fourier transform. The modes of oscillation can also be analyzed at this point.

Data

Length, l (m)	Angular Frequency, ω $\left(\frac{rad}{s}\right)$
0.3125	162
0.400	98
0.499	64.4
0.6065	43.75
0.699	32.2
0.7665	27.55
0.869	21.43

Data Summary



The dynamic flexural rigidity can be calculated from A in the fit from:

$$K = \rho\left(\frac{A}{(\beta_1 L)^2}\right)^2 \qquad \textbf{(EQ.19)}$$

Static flexural rigidity can be calculated from the dimensions of the bar, and the known Young's modulus for aluminum:

Static K	Dynamic K
4.33	4.30

The results between the static and dynamic flexural rigidity are very close. The analysis could be extended to different size and material cantilevers to see if results are consistent.

References

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